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Existence of Positive Solutions to Some Singular and Nonsingular Second Order Boundary Value Problems

D. O'REGAN*

School of Mathematics, Trinity College, Dublin 2, Ireland

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The topological transversality theorem is employed to study existence of solutions to a variety of singular and nonsingular second order boundary value problems of the form $y'' + f(t, y, y') = 0$, $0 < t < 1$. Here f may be singular at either $t = 0$, $t = 1$, $y = 0$, and/or $y' = 0$. The principal tools are the Arzelà–Ascoli theorem, essential maps, and a priori bounds on solutions. © 1990 Academic Press, Inc.

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This paper presents existence results for solutions to nonsingular and singular second order boundary value problems of the form

$$\begin{aligned} &y'' + f(t, y, y') = 0, \qquad 0 < t < 1 \\ &y \in B. \end{aligned} \tag{1.1}$$

Here B will denote either

1. $y(0) = a \geqslant 0$, $y'(1) = b \geqslant 0$,
2. $y(1) = a \geqslant 0$, $y'(0) = b \leqslant 0$, or
3. $y(0) = a \geqslant 0$, $y(1) = b \geqslant 0$.

The paper is divided into four main parts. In the first part we allow f to be singular at either $t = 0$ and/or $t = 1$. Problems of this form have been examined by many authors (see [2, 6, 7, 12], for example), however, our nonlinear term f does not satisfy the conditions in the above-mentioned papers so our results are new and complement those in the above references. For the second part of the paper we allow f to be singular at either $t = 0$, $t = 1$, and/or $y = 0$. Here f will have no dependence on y'. This

* On leave from University of Idaho, Moscow, ID 83843.

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singular problem has become quite popular in the last seven years or so; see [5, 9, 11, 3], for example. In the first three references the authors discuss specific problems and obtain existence theorems for such problems. However, in [3], with B denoting (3) and $a=b=0$, a more general problem was discussed and we in fact improve considerably the results of that paper; we will in fact only need about half of their assumptions. Thus our paper covers many new examples not treated by the above references and the results of the two papers are complementary. The last two parts of the paper are devoted to a relatively new and unexplored area of singular second order boundary value problems. The third part deals again with the case when f is singular at either $t=0$, $t=1$, and/or $y=0$ but f now has a y' dependence. Finally, we discuss in the last section problems where f is singular at either $t=0$, $t=1$, $y=0$, and/or $y'=0$.

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We begin this section by establishing existence of positive solutions on $[0, 1]$ to

$$\begin{aligned} y''+\psi(t)\, f(t, y, y')=0, \qquad 0<t<1 \\ y(0)=a>0 \\ y'(1)=b>0, \end{aligned} \tag{2.1}$$

where f satisfies the conditions:

f is continuous on $[0, 1] \times (0, \infty) \times (0, \infty)$ with $\lim_{y\to 0^+} f(t, y, p) = \infty$ for each $(t, p) \in [0, 1] \times (-\infty, \infty)/\{0\}$ and $\lim_{p\to 0^+} f(t, y, p)=\infty$ for each $(t, y) \in [0, 1] \times (0, \infty)$ (2.2)

$0 < f(t, y, p) \leqslant g(y)\,\phi(p)$ on $[0, 1] \times (0, \infty) \times (0, \infty)$, where g and $\phi>0$ are continuous and nonincreasing on $(0, \infty)$. (2.3)

In addition assume ψ satisfies

$1/\psi\colon [0, 1] \to [0, \infty)$ is continuous with $\psi>0$ on $(0, 1)$ and $\int_0^1 \psi(s)\, ds < \infty$. (2.4)

Now by a solution to (2.1) we mean a function $y \in C^1[0, 1] \cap C^2(0, 1)$ that satisfies the differential equation and the boundary conditions. Also if y is a solution to (2.1) then condition (2.2) implies $y>0$, $y'>0$ on $(0, 1)$

and as a result $y'' < 0$, on $(0, 1)$ so $y' > 0$ on $(0, 1)$, which in turn implies y is strictly increasing on $(0, 1)$, so in particular $y \geqslant a$ on $[0, 1]$.

THEOREM 2.1. *Suppose* (2.2), (2,3), *and* (2.4) *are satisfied. For* $\lambda \in [0, 1]$ *consider the family of problems*

$$\begin{aligned} y'' + \lambda\psi(t)\, f(t, y, y') = 0, \qquad 0 < t < 1 \\ y(0) = a \\ y'(1) = b. \end{aligned} \tag{2.1$_\lambda$}$$

Then there exist constants M_0, M_1, M_2 *independent of* λ *such that*

$$a \leqslant y(t) \leqslant M_0, \qquad b \leqslant y'(t) \leqslant M_1 \qquad \text{for} \quad t \in [0, 1]$$

and

$$-M_2 \leqslant y''(t)/\psi(t) \leqslant 0 \qquad \text{for} \quad t \in (0, 1)$$

for each solution y *to* $(2.1)_\lambda$.

Proof. Let y be a solution to $(2.1)_\lambda$. Then clearly $y(t) \geqslant a$, $y'(t) \geqslant b$ for $t \in [0, 1]$. We also have $y'' + \lambda\psi(t)\,\phi(y')\,g(y) \geqslant y'' + \lambda\psi(t)\,f(t, y, y') = 0$. Now this implies

$$0 \leqslant -y''(t) \leqslant \lambda g(y(t))\,\phi(y'(t))\,\psi(t) \leqslant g(a)\,\phi(b)\,\psi(t),$$

i.e.,

$$0 \leqslant \frac{-y''(t)}{\psi(t)} \leqslant g(a)\,\phi(b) \equiv M_2 \qquad \text{for} \quad t \in (0, 1).$$

Then

$$b \leqslant y'(t) = b - \int_t^1 y''(s)\,ds \leqslant b + M_2 \int_0^1 \psi(s)\,ds \equiv M_1$$

and

$$a \leqslant y(t) = a + \int_0^t y'(s)\,ds \leqslant a + M_1 \equiv M_0. \quad \blacksquare$$

Now Theorem 2.1 together with the Topological Transversality Theorem of Andrzej Granas [8] will be used to obtain our basic existence theorem.

We first, however, introduce the following notation. If $u \in C^1[0,1] \cap C^2(0,1)$ define

$$|u|_0 = \max_{t \in [0,1]} |u(t)| \qquad \text{and} \qquad |u|_2 = \max\left\{|u|_0, |u'|_0, \sup_{t \in (0,1)} |u''(t)/\psi(t)|\right\}.$$

Let $K = C(0,1)$ be the Banach space of functions ω continuous on $(0,1)$ and for which $\|\omega\|_\infty = \sup_{t \in (0,1)} |\omega(t)| < \infty$. Finally, let $K^2 = \{u \in C^2(0,1) \cap C^1[0,1] : |u|_2 < \infty\}$, which is a Banach space [8], and $K_B^2 = \{u \in K^2 : u(0) = a, u'(1) = b\}$ with $K_{B_0}^2 = \{u \in K^2 : u(0) = 0, u'(1) = 0\}$.

THEOREM 2.2. *Suppose* (2.2), (2.3), *and* (2.4) *are satisfied. Then a* $C^1[0,1] \cap C^2(0,1)$ *solution of* (2.1) *exists.*

Proof. Consider the family of problems

$$\begin{aligned} y'' + \lambda\psi(t)\bar{f}(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(0) &= a \\ y'(1) &= b, \end{aligned} \tag{2.5$_\lambda$}$$

where $\bar{f} > 0$ is any continuous extension of f from $y \geq a$ and $y' \geq b$. Now every solution v of $(2.5)_\lambda$ satisfies $v \geq a$, $v' \geq b$ and hence is a solution to $(2.1)_\lambda$. Also the conclusions of Theorem 2.1 remain valid for solutions to $(2.5)_\lambda$. Let

$$U = \{u \in K_B^2 : a/2 < |u|_0 < M_0 + 1, b/2 < |u'|_0 < M_1 + 1, \\ \|u''/\psi\|_\infty < M_2 + 1\}$$

and define mappings $F_\lambda : C^1[0,1] \to K$, $j : K_B^2 \to C^1[0,1]$, and $L : K_B^2 \to K$ by $F_\lambda v(t) = -\lambda \bar{f}(t, v(t), v'(t))$, $ju = u$, and $Lv(t) = v''(t)/\psi(t)$. Clearly F_λ is continuous from the continuity of $\bar{f}$. We next show that j is completely continuous. To see this suppose $\Omega \subset K_B^2$ is bounded, i.e., there exists $M < \infty$ such that $|y|_2 \leq M$ for all $y \in \Omega$. We claim $j\Omega$ is bounded and equicontinuous. Once the claim is established the Arzelà–Ascoli theorem implies j is completely continuous. Clearly $j\Omega$ is bounded and to show equicontinuity of $j\Omega$ consider $t, s \in [0,1]$. Then $|jy(t) - jy(s)| \leq M|t-s|$ clearly and in addition

$$\begin{aligned} |jy'(t) - jy'(s)| &\leq \left|\int_s^t |y''(z)|\,dz\right| \leq M\left|\int_s^t \psi(z)\,dz\right| \\ &= M\left|\int_s^{1/2} \psi(z)\,dz - \int_t^{1/2} \psi(z)\,dz\right|. \end{aligned}$$

Now since $\int_0^1 \psi(z)\,dz$ exists, $\int_t^{1/2} \psi(z)\,dz$ is continuous on $[0, 1]$ (in particular at 0 and 1 by the definition of an improper integral) so the equicontinuity of $j\Omega$ follows.

Finally, we claim that L^{-1} exists and is continuous. To show this define $N: K^2_{B_0} \to K$ by $Nu(t) = u''(t)/\psi(t)$. Now N^{-1} is a continuous linear operator by the Bounded Inverse Theorem. Thus L^{-1} exists and is given by

$$(L^{-1}g)(x) = a + bx + (N^{-1}g)(x)$$

and so is continuous. Now define the map $H_\lambda: \bar{U} \to K^2_B$ by $H_\lambda u = L^{-1}F_\lambda\, ju$. H_λ is a compact homotopy and the fixed points of H_λ are precisely the solutions of $(2.5)_\lambda$. Therefore H_λ is fixed point free on ∂U by the choice of U and Theorem 2.1. Finally, $H_0(u) = a + bx \in U$, the constant map sending each function to $a + bx$, is essential [8]. The Topological Transversality Theorem [8] implies that H_1 is essential, i.e., $(2.5)_1$ has a solution and therefore (2.1) has a solution. ∎

Remark. By making the change of variables $z(t) = y(1-t)$ and using the above result we can obtain a similar existence theorem for

$$\begin{aligned} y'' + \psi(t)\, f(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(1) &= a > 0 \\ y'(0) &= b < 0 \end{aligned}$$

with the only changes in assumptions (2.2), (2.3) being that f is continuous on $[0, 1] \times (0, \infty) \times (-\infty, 0)$ and $0 < f(t, y, p) \leqslant g(y)\, \phi(-p)$ on $[0, 1] \times (0, \infty) \times (-\infty, 0)$.

Finally, to conclude this section we establish existence of positive solutions on $[0, 1]$ to

$$\begin{aligned} y'' + \psi(t)\, f(t, y, y') &= 0, \qquad 0 < t < 1 \\ y(0) &= a > 0 \\ y(1) &= b > 0, \end{aligned} \tag{2.6}$$

where f satisfies the conditions

f is continuous on $[0, 1] \times (0, \infty) \times (-\infty, \infty)$ with $\lim_{y \to 0^+} f(t, y, p) = \infty$ for each $(t, p) \in [0, 1] \times (-\infty, \infty)$ (2.7)

$0 < f(t, y, p) \leqslant g(y)\, \phi(|p|)$ on $[0, 1] \times (0, \infty) \times (-\infty, \infty)$, where g is continuous and nonincreasing on $(0, \infty)$, $\phi > 0$ is continuous and nonincreasing on $[0, \infty)$. (2.8)

If y is a solution to (2.6) then $y''<0$ on $(0, 1)$ so y' is strictly decreasing on $(0, 1)$ and of course $y \geq \min\{a, b\}$ on $[0, 1]$.

THEOREM 2.3. *Suppose* (2.7), (2.8), *and* (2.4) *are satisfied. Then a* $C^1[0, 1] \cap C^2(0, 1)$ *solution of* (2.6) *exists.*

Proof. This follows from a slight modification of the proofs of Theorems 2.1 and 2.2. ∎

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In this section we again consider the problems in Section 2, where f may now be singular at $y=0$. In addition f will have no y' dependence; the more difficult situation when f has a y' dependence will be considered in Sections 4 and 5. To begin with we will establish the existence of positive solutions on $(0, 1]$ to

$$\begin{aligned} y'' + \psi(t)\, f(t, y) &= 0, \qquad 0<t<1 \\ y(0) &= 0 \\ y'(1) &= b \geq 0, \end{aligned} \tag{3.1}$$

where f and ψ satisfy

f is continuous on $[0, 1] \times (0, \infty)$ with $\lim_{y \to 0^+} f(t, y) = \infty$ for each $t \in [0, 1]$ and $0 < f(t, y) \leq g(y)$ on $[0, 1] \times (0, \infty)$, where g is continuous and nonincreasing on $(0, \infty)$. In addition $1/\psi \in C[0, 1]$ with $\psi > 0$ on $(0, 1)$. (3.2)

There exist $p>1$, $q>1$ with $1/p + 1/q = 1$ together with $\int_0^1 \psi^p(z)\, dz < \infty$ and $\int_0^1 g^q(u)\, du < \infty$. (3.3)

For each constant $M>0$ there exists $\eta(t)$ continuous and positive on $[0, 1]$ such that $f(t, y) \geq \eta(t)$ on $[0, 1] \times (0, M]$. (3.4)

To establish the existence of a solution to (3.1) we first consider for $n \in N^+ = \{1, 2, ...\}$ the problems

$$\begin{aligned} y'' + \psi(t)\, f(t, y) &= 0, \qquad 0<t<1 \\ y(0) &= 1/n \\ y'(1) &= b \geq 0 \end{aligned} \tag{3.1$_n$}$$

and then we will use a compactness argument. Now Theorem 2.2 implies that $(3.1)_n$ has a solution y_n for each n.

LEMMA 3.1. *Suppose* (3.2) *and* (3.3) *are satisfied. Then there are constants* M_0 *and* M_1 *independent of* n *such that*

$$1/n \leqslant |y|_0 \leqslant M_0, \qquad b \leqslant |y'|_0 \leqslant M_1$$

for each solution $y \in C^1[0,1] \cap C^2(0,1)$ *to* $(3.1)_n$.

Proof. Let y be a solution to $(3.1)_n$. Then clearly $y(t) \geqslant 1/n$, $y'(t) \geqslant b$ for $t \in [0,1]$. We also have $y'' + \lambda\psi(t)\, g(y) \geqslant 0$. Integrating from t to 1 yields

$$\begin{aligned} y'(t) &\leqslant b + \int_t^1 g(y(s))\, \psi(s)\, ds \leqslant g(y(t)) \int_t^1 \psi(s)\, ds + b \\ &\leqslant b + g(y(t)) \int_0^1 \psi(s)\, ds \leqslant g(y(t) - 1/n) \int_0^1 \psi(s)\, ds + b \end{aligned}$$

since g is nonincreasing. Let $M = \int_0^1 \psi(s)\, ds$ so integrate from 0 to t to obtain

$$\int_0^{y(t)-1/n} \frac{du}{Mg(u)+b} \leqslant 1.$$

Define $G(z) = \int_0^z (du/(Mg(u)+b))$ so G is an increasing map from $[0,\infty)$ onto $[0,\infty)$ and therefore has an increasing inverse G^{-1}. Thus it follows that

$$y(t) \leqslant G^{-1}(1) + 1/n \leqslant G^{-1}(1) + 1 \equiv M_0 \qquad \text{for} \quad t \in [0,1].$$

We have thus shown that

$$1/n \leqslant y(t) \leqslant M_0, \qquad t \in [0,1]$$

for any solution y to $(3.1)_n$. Now returning to the inequality $-y'' \leqslant \psi(t)\, g(y(t))$ and multiplying by $(y')^{1/q}$ yields

$$-(y')^{1/q}\, y'' \leqslant \psi(t)\, g(y(t))(y')^{1/q}.$$

Integrate from t to 1 and use Hölder's integral inequality to get

$$\begin{aligned} \frac{q}{1+q}[y'(t)]^{(1+q)/q} \leqslant{}& \frac{q}{1+q} b^{(1+q)/q} \\ &+ \left\{\int_t^1 \psi^p(s)\, ds\right\}^{1/p} \left\{\int_t^1 g^q(y(s))\, y'(s)\, ds\right\}^{1/q}, \end{aligned}$$

i.e.,

$$y'(t) \leqslant \left\{ b^{(q+1)/q} + \frac{q+1}{q} \left(\int_0^1 \psi^p(s)\, ds \right)^{1/p} \left(\int_0^{M_0} g^q(u)\, du \right)^{1/q} \right\}^{q/(q+1)} \equiv M_1.$$

We have shown that

$$b \leqslant y'(t) \leqslant M_1, \qquad t \in [0, 1]$$

for any solution y to $(3.1)_n$. ∎

The results of Section 2 together with the above lemma and the Arzelà–Ascoli theorem will immediately yield

THEOREM 3.1. *Suppose* (3.2), (3.3), *and* (3.4) *are satisfied. Then a* $C[0, 1] \cap C^2(0, 1)$ *solution of* (3.1) *exists.*

Proof. Now Theorem 2.2 implies that $(3.1)_n$ has a solution y_n for each n. Moreover by Lemma 3.1 there are constants M_0 and M_1 independent of n such that

$$1/n \leqslant y_n(t) \leqslant M_0, \qquad b \leqslant y_n'(t) \leqslant M_1 \qquad \text{for} \quad t \in [0, 1].$$

Now the Arzelà–Ascoli theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some continuous function y, i.e., $|y_{n'} - y|_0 \to 0$ for some $y \in C[0, 1]$. Clearly $y \geqslant 0$ on $[0, 1]$. In fact $y > 0$ on $(0, 1]$; to see this recall from (3.4) that $-y_n''(t) \geqslant \psi(t)\,\eta(t)$ so integrating twice yields

$$y_n(t) \geqslant 1/n + bt + \int_0^t s\psi(s)\,\eta(s)\, ds + t \int_t^1 \psi(s)\,\eta(s)\, ds.$$

Now $y_{n'}$ satisfies the integral equation

$$y_{n'}(t) = y_{n'}(1) + b(t-1) + \int_t^1 (t-s)\, f(s, y_{n'}(s))\, \psi(s)\, ds$$

so for $t \in (0, 1]$ and $s \in [t, 1]$ we have that $f(s, y_{n'}(s)) \to f(s, y(s))$ uniformly since f is uniformly continuous on compact subsets of $[0, 1] \times (0, M_0]$. Thus letting $n' \to \infty$ yields

$$y(t) = y(1) + b(t-1) + \int_t^1 (t-s)\, f(s, y(s))\, \psi(s)\, ds.$$

From this integral equation we see that $y \in C^2(0, 1)$ and $y''(t) = -f(t, y(t))\,\psi(t)$. ∎

Remark. Similarly we have corresponding results for the problem

$$\begin{aligned} y''+\psi(t)\, f(t, y)&=0, \qquad 0<t<1 \\ y(1)&=0 \\ y'(0)&=b\leqslant 0. \end{aligned}$$

Finally, we turn our attention to

$$\begin{aligned} y''+\psi(t)\, f(t, y)&=0, \qquad 0<t<1 \\ y(0)&=0 \\ y(1)&=0, \end{aligned} \tag{3.5}$$

where (3.2), (3.3), and (3.4) are again satisfied. As before we first consider

$$\begin{aligned} y''+\psi(t)\, f(t, y)&=0, \qquad 0<t<1 \\ y(0)&=1/n \\ y(1)&=1/n. \end{aligned} \tag*{$(3.5)_n$}$$

Now Theorem 2.3 implies that $(3.5)_n$ has a solution y_n for each n.

LEMMA 3.2. *Suppose* (3.2) *and* (3.3) *are satisfied. Then there are constants M_0 and M_1 independent on n such that*

$$1/n\leqslant |y|_0\leqslant M_0, \qquad |y'|_0\leqslant M_1$$

for each solution $y\in C^1[0, 1]\cap C^2(0, 1)$ *to* $(3.5)_n$.

Proof. Let y be a solution to $(3.5)_n$. Then clearly $y(t)\geqslant 1/n$ for $t\in[0, 1]$. Let $y_{\max}$ be the maximum of $y(t)$ on $[0, 1]$. If the maximum of $y(t)$ occurs at the endpoints then $y\equiv 1/n$. Now suppose $y_{\max}$ occurs at $t_0\in(0, 1)$, so $y'(t_0)=0$. Integrate from t_0 to $t>t_0$ to obtain

$$\begin{aligned} -y'(t)&\leqslant \int_{t_0}^{t} g(y(s))\,\psi(s)\,ds\leqslant g(y(t))\int_{t_0}^{t}\psi(s)\,ds \\ &\leqslant g(y(t)-1/n)\int_0^1 \psi(s)\,ds \end{aligned}$$

since g is nonincreasing. Divide by $g(y(t)-1/n)$ and integrating from t_0 to 1 will give

$$\int_0^{y_{\max}-1/n}\frac{du}{g(u)}\leqslant\int_0^1\psi(s)\,ds.$$

Define $H(z) = \int_0^z (du/g(u))$ and H has an increasing inverse H^{-1}. Hence

$$y_{\max} \leqslant H^{-1}\left(\int_0^1 \psi(s)\,ds\right) + 1/n \leqslant H^{-1}\left(\int_0^1 \psi(s)\,ds\right) + 1 \equiv M_0.$$

We have shown that

$$1/n \leqslant y(t) \leqslant M_0, \qquad t \in [0, 1]$$

for any solution y to $(3.5)_n$. Now since $y(0) = y(1) = 1/n$ there exists $\xi \in (0, 1)$ with $y'(\xi) = 0$. For $t < \xi$ we have $y' > 0$ so

$$-(y')^{1/q}\, y'' \leqslant \psi(t)\, g(y)(y')^{1/q}$$

and integration from t to ξ with Hölder's integral inequality yields

$$\begin{aligned} y'(t) &\leqslant \left\{\frac{q+1}{q}\left(\int_0^1 \psi^p(s)\,ds\right)^{1/p}\left(\int_0^{M_0} g^q(u)\,du\right)^{1/q}\right\}^{q/(q+1)} \\ &\equiv M_1 \qquad \text{for} \quad t \in [0, \xi]. \end{aligned}$$

On the other hand, for $t > \xi$ we have

$$-(-y')^{1/q}\, y'' \leqslant \psi(t)\, g(y)(-y')^{1/q}$$

and integration from ξ to t yields

$$\begin{aligned} -y'(t) &\leqslant \left\{\frac{q+1}{q}\left(\int_0^1 \psi^p(s)\,ds\right)^{1/p}\left(\int_0^{M_0} g^q(u)\,du\right)^{1/q}\right\}^{q/(q+1)} \\ &\equiv M_1 \qquad \text{for} \quad t \in [\xi, 1]. \end{aligned}$$

Thus

$$|y'(t)| \leqslant M_1, \qquad t \in [0, 1]$$

for any solution y to $(3.5)_n$. ∎

THEOREM 3.2. *Suppose* (3.2), (3.3), *and* (3.4) *are satisfied. Then a* $C[0, 1] \cap C^2(0, 1)$ *solution of* (3.5) *exists.*

Proof. This follows from a slight modification of the proof of Theorem 3.1. ∎

Remark. We can obtain similar results for problems of the form

$$\begin{aligned} y'' + \psi(t)\, f(t, y) &= 0, \qquad 0 < t < 1 \\ y(0) &= 0 \\ y(1) &= b > 0 \end{aligned}$$

and

$$
\begin{aligned}
y'' + \psi(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\
y(0) = a > 0 \\
y(1) = 0.
\end{aligned}
$$

In some practical situations condition (3.3) is not satisfied. However, the techniques in this section illustrate a method for obtaining existence of a solution even in these cases. The following example is motivated by the Blasius equation, which arises in Newtonian fluid theory; see [5].

EXAMPLE. Consider

$$
\begin{aligned}
y'' + \psi(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\
y(0) = 0 \\
y'(1) = 0,
\end{aligned}
\tag{3.6}
$$

where (3.2) is satisfied with $g(y) = y^{-1}$. In addition assume $\int_0^1 \psi(s)\, ds < \infty$ and also that (3.4) is satisfied.

To establish existence of a solution to (3.6) we first consider for $n \in N^+ = \{1, 2, ...\}$

$$
\begin{aligned}
y'' + \psi(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\
y(0) = 1/n \\
y'(1) = 0.
\end{aligned}
\tag*{$(3.6)_n$}
$$

Now Theorem 2.2 implies that $(3.6)_n$ has a solution y_n for each n. We next claim that there are constants M_0 and M_1 independent of n such that

$$1/n \leqslant |y|_0 \leqslant M_0 \qquad \text{and} \qquad \|y'\|_{L^2} = \left\{ \int_0^1 [y'(s)]^2\, ds \right\}^{1/2} \leqslant M_1.$$

To see this let y be a solution to $(3.6)_n$. Then clearly $y(t) \geqslant 1/n$, $y'(t) \geqslant 0$ for $t \in [0, 1]$ and in addition by exactly the same argument in Lemma 3.1 there exists a constant M_0 independent of n such that $1/n \leqslant |y|_0 \leqslant M_0$. In addition $-y'' \leqslant \psi(t)\, y^{-1}$ so $-yy'' \leqslant \psi(t)$ and integrating from 0 to 1 yields

$$\frac{y'(0)}{n} + \int_0^1 [y'(s)]^2\, ds \leqslant \int_0^1 \psi(s)\, ds.$$

Now since $y'(0) \geqslant 0$ we have

$$\|y'\|_{L^2} \leqslant \left\{ \int_0^1 \psi(s)\, ds \right\}^{1/2} \equiv M_1.$$

We have shown that $(3.6)_n$ has a solution y_n for each n and that there are constant M_0 and M_1 independent of n such that

$$1/n \leqslant |y_n|_0 \leqslant M_0, \qquad \|y_n'\|_{L^2} \leqslant M_1.$$

It follows that $\{y_n\}$ is uniformly bounded and equicontinuous (by Hölder's integral inequality with $p=q=2$) on $[0, 1]$. Now the Arzelà–Ascoli theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some continuous function y. Now clearly $y \geqslant 0$ and in fact $y > 0$ on $(0, 1]$ since condition (3.4) is satisfied. Also $y_{n'}$ satisfies the integral equation

$$y_{n'}(t) = y_{n'}(1) + \int_t^1 (t-s)\, f(s, y_{n'}(s))\, \psi(s)\, ds$$

and letting $n' \to \infty$ (see Theorem 3.1) guarantees the existence of a $C[0, 1] \cap C^2(0, 1)$ solution to (3.6).

Motivated from the last example we can obtain alternative existence theorems to those in this section. The behavior of g will, in general, determine which existence theorem to use. Again consider problems of the form (3.1) with assumptions (2.4), (3.2), and (3.4) being satisfied. In addition suppose (3.3) is replaced by

$$yg(y) \text{ is nondecreasing on } (0, \infty). \tag{3.7}$$

To establish the existence of a solution to (3.1) let y be a solution to $(3.1)_n$; the existence of such a solution follows automatically from Theorem 2.2. Then clearly $y(t) \geqslant 1/n$, $y'(t) \geqslant b$ for $t \in [0, 1]$. Also by the argument in Lemma 3.1 there exists a constant M_0 independent of n such that $1/n \leqslant |y|_0 \leqslant M_0$. In addition $-y'' \leqslant g(y)\, \psi(t)$ yields

$$-yy'' \leqslant yg(y)\, \psi(t) \leqslant M_0\, g(M_0)\, \psi(t)$$

and integrating from 0 to 1 gives

$$-by(1) + \frac{y'(0)}{n} + \int_0^1 [y'(s)]^2\, ds \leqslant M_0\, g(M_0) \int_0^1 \psi(s)\, ds,$$

i.e.,

$$\|y'\|_{L^2} \leqslant \left\{ M_0\, g(M_0) \int_0^1 \psi(s)\, ds + bM_0 \right\}^{1/2} \equiv M_1.$$

We have shown that $(3.1)_n$ has a solution y_n for each n and that there are constants M_0 and M_1 independent of n such that

$$1/n \leqslant |y_n|_0 \leqslant M_0, \qquad \|y_n'\|_{L^2} \leqslant M_1.$$

Thus we obtain via the ideas of Theorem 3.1 and the last example:

THEOREM 3.3. *Suppose* (2.4), (3.2), (3.4), *and* (3.7) *are satisfied. Then a* $C[0, 1] \cap C^2(0, 1)$ *solution of* (3.1) *exists.*

Remark. We can obtain similar results for problems of the form (3.5).

4

In this section we consider problems where f, which now has a y' dependence, may be singular at $y=0$ but f is *not* singular at $y'=0$. In particular we consider problems of the form

$$\begin{aligned} y'' + \psi(t)\, f(t, y, y') = 0, &\qquad 0<t<1 \\ y(0) = 0 & \\ y'(1) = b > 0, & \end{aligned} \tag{4.1}$$

where $1/\psi \in C[0, 1]$ with $\psi > 0$ on $(0, 1)$. In addition assume (2.2), (2.3), and (2.4) are satisfied. Moreover suppose the following also holds:
There exists $r > 1$ with

$$\int_0^1 g^r(bt)\, \psi^r(t)\, dt < \infty. \tag{4.2}$$

To establish existence of a solution to (4.1) we first consider for $n \in N^+ = \{1, 2, ...\}$ the problems

$$\begin{aligned} y'' + \psi(t)\, f(t, y, y') = 0, &\qquad 0<t<1 \\ y(0) = 1/n & \\ y'(1) = b > 0. & \end{aligned} \tag*{$(4.1)_n$}$$

Now Theorem 2.2 implies that $(4.1)_n$ has a solution y_n for each n.

LEMMA 4.1. *Suppose* (2.2), (2.3), (2.4), *and* (4.2) *are satisfied. Then there exist constants* M_0, M_1, *and* M_2 *independent of* n *such that*

$$1/n \leqslant |y|_0 \leqslant M_0, \qquad b \leqslant |y'|_0 \leqslant M_1,$$

$$\|y''\|_{L^r} = \left\{\int_0^1 |y''(z)|^r \, dz\right\}^{1/r} \leqslant M_2$$

for each solution $y \in C^1[0, 1] \cap C^2(0, 1)$ *to* $(4.1)_n$.

Proof. Let y be a solution to $(4.1)_n$. Then clearly $y(t) \geqslant 1/n$, $y'(t) \geqslant b$ for $t \in [0, 1]$; in fact $y(t) \geqslant bt$ for $t \in [0, 1]$. We also have

$$-y'' \leqslant g(y)\,\phi(y')\,\psi(t) \leqslant g(y)\,\phi(b)\,\psi(t)$$

since ϕ is nonincreasing. Then

$$\|y''\|_{L^r} = \left\{\int_0^1 |y''(s)|^r \, ds\right\}^{1/r} \leqslant \left\{\int_0^1 g^r(y(t))\,\phi^r(y'(t))\,\psi^r(t)\,dt\right\}^{1/r}$$

$$\leqslant \phi(b)\left\{\int_0^1 g^r(bt)\,\psi^r(t)\,dt\right\}^{1/r} \equiv M_2.$$

Finally, integration and Hölder's integral inequality yield M_0 and M_1. ∎

THEOREM 4.1. *Suppose* (2.2), (2.3), (2.4), *and* (4.2) *are satisfied. Then a* $C^1[0, 1] \cap C^2(0, 1)$ *solution of* (4.1) *exists.*

Proof. Theorem 2.2 implies that $(4.1)_n$ has a solution y_n for each n. Moreover by Lemma 4.1 there are constants M_0, M_1, and M_2 independent of n such that

$$1/n \leqslant |y_n|_0 \leqslant M_0, \qquad b \leqslant |y_n'|_0 \leqslant M_1, \qquad \|y_n''\|_{L^r} \leqslant M_2.$$

Now the Arzelà–Ascoli theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some continuously differentiable function y, i.e., $|y_{n'} - y|_1 \to 0$ for some $y \in C^1[0, 1]$; here $|u|_1 = \max\{|u|_0, |u'|_0\}$ for $u \in C^1[0, 1]$. Clearly $y \geqslant 0$ and $y' \geqslant b$ on $[0, 1]$. In fact $y > 0$ on $(0, 1]$ since $b > 0$. Now $y_{n'}$ satisfies the integral equation

$$y_{n'}(t) = y_{n'}(1) + b(t-1) + \int_t^1 (t-s)\,f(s, y_{n'}(s), y_{n'}'(s))\,\psi(s)\,ds$$

so for $t \in (0, 1]$ and $s \in [t, 1]$ we have that $f(s, y_{n'}(s), y_{n'}'(s)) \to$

$f(s, y(s), y'(s))$ uniformly since f is uniformly continuous on compact subsets of $[0, 1] \times (0, M_0] \times [b, M_1]$. Thus letting $n' \to \infty$ yields

$$y(t) = y(1) + b(t-1) + \int_t^1 (t-s)\, f(s, y(s), y'(s))\, \psi(s)\, ds.$$

From the integral equation we see that $y \in C^2(0, 1)$ and $y''(t) = -f(t, y(t), y'(t))\, \psi(t)$. ∎

Remark. Similarly we have corresponding results for the problem

$$\begin{aligned} y'' + \psi(t)\, f(t, y, y') = 0, \qquad 0 < t < 1 \\ y(1) = 0 \\ y'(0) = b < 0. \end{aligned}$$

Remark. It should also be noted here that the ideas of this section could be applied to obtain alternative existence theorems to those in Section 3. Consider, for example,

$$\begin{aligned} y'' + \psi(t)\, f(t, y) = 0, \qquad 0 < t < 1 \\ y(0) = 0 \\ y'(1) = b > 0. \end{aligned} \tag{4.3}$$

Suppose (3.2) and (3.4) are satisfied. In addition suppose (3.3) is replaced by (4.2) with $r = 1$. Then a $C[0, 1] \cap C^2(0, 1)$ solution of (4.3) exists; to see this let y be a solution to $(3.1)_n$. Now $|y'(t)| \leqslant b + \int_t^1 |y''(z)|\, dz \leqslant b + \int_t^1 g(y(z))\, \psi(z)\, dz \leqslant b + \int_t^1 g(bz)\, \psi(z)\, dz \equiv M_1$ and by integration $y(t) \leqslant 1/n + M_1 \leqslant 1 + M_1 \equiv M_0$ for $t \in [0, 1]$. Now the existence of a $C[0, 1] \cap C^2(0, 1)$ solution to (4.3) follows from Theorem 3.1.

Remark. It is possible to extend the class of functions f that we have previously considered. Again consider problems of the form (4.1) with assumptions (2.4) and (4.2) being satisfied. In addition suppose the following also hold:

> f is continuous on $[0, 1] \times (0, \infty) \times (-\infty, \infty)$ with $f \geqslant 0$ on $[0, 1] \times (0, \infty) \times (-\infty, \infty)$ and $\lim_{y \to 0^+} f(t, y, p) = \infty$ for each $(t, p) \in [0, 1] \times (-\infty, \infty)/\{0\}$ (4.4)
>
> $0 < f(t, y, p) \leqslant g(y)\, \phi(p)$ on $[0, 1] \times (0, \infty) \times [b, \infty)$, where g is continuous and nonincreasing on $(0, \infty)$ and $\phi > 0$ is continuous on $[b, \infty)$ with $\{\int_0^1 g^r(bs)\, \psi^r(s)\, ds\}^{1/r} < \int_b^\infty (du/\phi(u))$. (4.5)

To establish the existence of a solution to (4.1) let y be a solution to $(4.1)_n$. Then clearly $y(t) \geqslant 1/n$, $y'(t) \geqslant b$ for $t \in [0, 1]$; in fact $y(t) \geqslant bt$ for $t \in [0, 1]$. We also have

$$-y'' \leqslant g(y)\,\phi(y')\,\psi(t).$$

Now divide by $\phi(y')$ and integrate from t to 1 to obtain

$$\int_b^{y'(t)} \frac{du}{\phi(u)} \leqslant \int_t^1 g(y(s))\,\psi(s)\,ds \leqslant \int_t^1 g(bs)\,\psi(s)\,ds$$
$$\leqslant \left\{\int_0^1 g^r(bs)\,\psi^r(s)\,ds\right\}^{1/r}.$$

Let $I(z) = \int_b^z (du/\phi(u))$ so

$$y'(t) \leqslant I^{-1}\left(\left\{\int_0^1 g^r(bs)\,\psi^r(s)\,ds\right\}^{1/r}\right) \equiv M_1, \qquad t \in [0, 1]$$

and thus $y(t) \leqslant M_1 + 1/n \leqslant M_1 + 1 \equiv M_0$ for $t \in [0, 1]$. In addition for $t \in (0, 1)$

$$\frac{-y''(t)}{\psi(t)} \leqslant g(y(t))\,\phi(y'(t)) \leqslant g(1/n) \sup_{[b, M_1]} \phi(p) \equiv \tilde{M}_2.$$

Thus $(4.1)_n$ has a solution y_n for each n by the ideas of Theorem 2.2. In addition we have shown that there are constants M_0 and M_1 (independent of n) such that

$$1/n \leqslant |y_n|_0 \leqslant M_0, \qquad b \leqslant |y_n'|_0 \leqslant M_1.$$

We also have

$$-y''(t) \leqslant g(y(t))\,\phi(y'(t))\,\psi(t) \leqslant Kg(y(t))\,\psi(t),$$

where $K = \sup_{[b, M_1]} \phi(p)$. Thus

$$\|y''\|_{L^r} \leqslant K\left(\int_0^1 g^r(bs)\,\psi^r(s)\,ds\right)^{1/r} \equiv M_2.$$

Now the Arzelà–Ascoli theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some continuously differentiable function y and via ideas of Theorem 4.1 we obtain

THEOREM 4.2. *Suppose* (2.4), (4.2), (4.4), *and* (4.5) *are satisfied. Then a* $C^1[0, 1] \cap C^2(0, 1)$ *solution of* (4.1) *exists.*

5

To conclude this paper we consider boundary value problems where f may be singular either at $y=0$ and/or $y'=0$; here f has a y' dependence. We first discuss the problem where f may be singular at $y'=0$ but f is *not* singular at $y=0$. In particular we consider

$$\begin{aligned} y''+\psi(t)\,f(t,y,y')=0, \qquad 0<t<1 \\ y(0)=a>0 \\ y'(1)=0. \end{aligned} \tag{5.1}$$

Assume (2.2), (2.3), and (2.4) are satisfied. In addition assume the following also hold:

> For constants $K>a$, $L>0$ there exists $\eta(t)$ continuous and positive on $[0,1]$ such that $f(t,y,p)\geq\eta(t)$ on $[0,1]\times(a,K]\times(0,L]$. (5.2)

There exists $r>1$ with

$$\int_0^1 \phi^r\left(\int_t^1 \psi(s)\,\eta(s)\,ds\right)\psi^r(t)\,dt<\infty. \tag{5.3}$$

To establish existence of a solution to (5.1) we first consider for $n\in N^+=\{1,2,\ldots\}$

$$\begin{aligned} y''+\psi(t)\,f(t,y,y')=0, \qquad 0<t<1 \\ y(0)=a>0 \\ y'(1)=1/n. \end{aligned} \tag{5.1$_n$}$$

LEMMA 5.1. *Suppose* (2.2), (2.3), (2.4), (5.2), *and* (5.3) *are satisfied. Then there exist constants* M_0, M_1, *and* M_2 *independent of* n *such that*

$$a\leq|y|_0\leq M_0, \qquad 1/n\leq|y'|_0\leq M_1, \qquad \|y''\|_{L^r}\leq M_2$$

for each solution $y\in C^1[0,1]\cap C^2(0,1)$ *to* $(5.1)_n$.

Proof. Let y be a solution to $(5.1)_n$. Then clearly $y(t)\geq a$, $y'(t)\geq 1/n$ for $t\in[0,1]$. We also have

$$-y''\leq g(y)\,\phi(y')\,\psi(t)\leq g(a)\,\phi(y'-1/n)\,\psi(t)$$

since ϕ and g are nonincreasing. Integrate from t to 1 to obtain

$$\int_0^{y'(t)-1/n} \frac{du}{\phi(u)} \leqslant g(a) \int_0^1 \psi(s)\, ds.$$

Let $M = \int_0^1 \psi(s)\, ds$ and define $J(z) = \int_0^z (du/\phi(u))$. Now J is an increasing map from $[0, \infty)$ onto $[0, \infty)$ and therefore has an increasing inverse J^{-1}. Thus we have

$$y'(t) \leqslant J^{-1}(Mg(a)) + 1/n \leqslant J^{-1}(Mg(a)) + 1 \equiv M_1$$

for $t \in [0, 1]$. Now integrating from 0 to t yields

$$y(t) \leqslant M_1 + a \equiv M_0 \qquad \text{for} \quad t \in [0, 1].$$

Thus we have shown that

$$a \leqslant y(t) \leqslant M_0, \qquad 1/n \leqslant y'(t) \leqslant M_1, \qquad t \in [0, 1]$$

for any solution y to $(5.1)_n$. Now assumption (5.2) implies that $-y''(t) \geqslant \psi(t)\, \eta(t)$ so $y'(t) \geqslant \int_t^1 \psi(s)\, \eta(s)\, ds$, which yields

$$\begin{aligned}\|y''\|_{L^r} &= \left\{\int_0^1 |y''(s)|^r\, ds\right\}^{1/r} \leqslant \left\{\int_0^1 g^r(y(t))\, \phi^r(y'(t))\, \psi^r(t)\, dt\right\}^{1/r} \\ &\leqslant g(a) \left\{\int_0^1 \phi^r \left(\int_t^1 \psi(s)\, \eta(s)\, ds\right) \psi^r(t)\, dt\right\}^{1/r} \equiv M_2. \quad \blacksquare\end{aligned}$$

THEOREM 5.1. *Suppose* (2.2), (2.3), (2.4), (5.2), *and* (5.3) *are satisfied. Then a* $C^1[0, 1] \cap C^2(0, 1)$ *solution of* (5.1) *exists.*

Proof. Theorem 2.2 implies that $(5.1)_n$ has a solution y_n for each n. Moreover by Lemma 5.1 there are constants M_0, M_1, and M_2 independent of n such that

$$a \leqslant |y_n|_0 \leqslant M_0, \qquad 1/n \leqslant |y_n'|_0 \leqslant M_1, \qquad \|y_n''\|_{L^r} \leqslant M_2.$$

Now the Arzelà–Ascoli theorem guarantees the existence of a subsequence $y_{n'}$ converging uniformly on $[0, 1]$ to some continuously differentiable function y. Clearly $y \geqslant a$ and $y' \geqslant 0$ on $[0, 1]$. In fact $y' > 0$ on $[0, 1)$ since condition (5.2) is satisfied. Now $y_{n'}$ satisfies the integral equation

$$y_{n'}(t) = a + t y_{n'}'(0) + \int_0^t (s - t)\, f(s, y_{n'}(s), y_{n'}'(s))\, \psi(s)\, ds$$

so for $t \in [0, 1)$ and $s \in [0, t]$ we have that $f(s, y_{n'}(s), y'_{n'}(s)) \to f(s, y(s), y'(s))$ uniformly since f is uniformly continuous on compact subsets of $[0, 1] \times [a, M_0] \times (0, M_1]$. Thus letting $n' \to \infty$ yields

$$y(t) = a + y'(0)t + \int_0^t (s - t) f(s, y(s), y'(s)) \psi(s) \, ds.$$

From the integral equation we see that $y \in C^2(0, 1)$ and $y''(t) = -f(t, y(t), y'(t)) \psi(t)$. ∎

Finally, it remains to discuss the problem where f may be singular at either $y = 0$ and/or $y' = 0$. In particular we consider

$$\begin{aligned} y'' + \psi(t) f(t, y, y') = 0, &\qquad 0 < t < 1 \\ y(0) = 0 & \\ y'(1) = 0. & \end{aligned} \tag{5.4}$$

Assume (2.2), (2.3), and (3.3) are satisfied. In addition assume the following also hold:

For constants $K > 0$, $L > 0$ there exists $\eta(t)$ continuous and positive on $[0, 1]$ such that $f(t, y, p) \geq \eta(t)$ on $[0, 1] \times (0, K] \times (0, L]$. (5.5)

There exists $r > 1$ with

$$\int_0^1 g^r \left(t \int_t^1 \psi(s) \eta(s) \, ds + \int_0^t s \psi(s) \eta(s) \, ds \right) \times \phi^r \left(\int_t^1 \psi(s) \eta(s) \, ds \right) \psi^r(t) \, dt < \infty. \tag{5.6}$$

To establish existence of a solution to (5.4) we first consider for $n \in N^+$

$$\begin{aligned} y'' + \psi(t) f(t, y, y') = 0, &\qquad 0 < t < 1 \\ y(0) = 1/n & \\ y'(1) = 1/n, & \end{aligned} \tag{5.4_n}$$

where $1/\psi \in C[0, 1]$ with $\psi > 0$ on $(0, 1)$.

LEMMA 5.2. *Suppose* (2.2), (2.3), (3.3), (5.5), *and* (5.6) *are satisfied. Then there exist constants* M_0, M_1, *and* M_2 *independent of n such that*

$$1/n \leq |y|_0 \leq M_0, \qquad 1/n \leq |y'|_0 \leq M_1, \qquad \|y''\|_{L^r} \leq M_2$$

for each solution $y \in C^1[0, 1] \cap C^2(0, 1)$ *to* $(5.4)_n$.

Proof. Let y be a solution to $(5.4)_n$. Then clearly $y(t) \geqslant 1/n$, $y'(t) \geqslant 1/n$ for $t \in [0, 1]$. Also we have

$$-y'' \leqslant g(y)\,\phi(y')\,\psi(t) \leqslant g(y)\,\phi(y'-1/n)\,\psi(t)$$

and so integrating from t to 1 yields

$$\int_0^{y'(t)-1/n} \frac{du}{\phi(u)} \leqslant \int_t^1 g(y(s))\,\psi(s)\,ds \leqslant g(y(t)) \int_t^1 \psi(s)\,ds$$
$$\leqslant g(y(t)-1/n) \int_0^1 \psi(s)\,ds.$$

Let $M = \int_0^1 \psi(s)\,ds$ and $J(z) = \int_0^z (du/\phi(u))$. Hence

$$y'(t) \leqslant J^{-1}(Mg(y(t)-1/n)) + 1/n \leqslant J^{-1}(Mg(y(t)-1/n)) + 1.$$

Divide by $J^{-1}(Mg(y(t)-1/n)) + 1$ and integrate from 0 to t to obtain

$$\int_0^{y(t)-1/n} \frac{du}{J^{-1}(Mg(u)) + 1} \leqslant 1.$$

Define

$$G(z) = \int_0^z \frac{du}{J^{-1}(Mg(u)) + 1}$$

and since $1/(J^{-1}(Mg(u)) + 1)$ is an increasing function, G is an increasing map from $[0, \infty)$ onto $[0, \infty)$ and therefore has an increasing inverse G^{-1}. Thus it follows that

$$y(t) \leqslant G^{-1}(1) + 1/n \leqslant G^{-1}(1) + 1 \equiv M_0 \qquad \text{for} \quad t \in [0, 1].$$

Now returning to the inequality $-y'' \leqslant g(y)\,\phi(y')\,\psi(t)$ and multiplying by $(y'-1/n)^{1/q}$ yields

$$-(y'-1/n)^{1/q}\,y'' \leqslant \psi(t)\,g(y)\,\phi(y')(y'-1/n)^{1/q}$$
$$\leqslant \psi(t)\,g(y)\,\phi(y'-1/n)(y')^{1/q}$$

since ϕ is nonincreasing. Integrate from t to 1 and use Hölder's inequality to get

$$\int_0^{y'(t)-1/n} \frac{u^{1/q}}{\phi(u)}\,du \leqslant \left\{\int_t^1 \psi^p(s)\,ds\right\}^{1/p} \left\{\int_t^1 g^q(y(s))\,y'(s)\,ds\right\}^{1/q}$$
$$\leqslant \left\{\int_0^1 \psi^p(s)\,ds\right\}^{1/p} \left\{\int_0^{M_0} g^q(u)\,du\right\}^{1/q} \equiv \tilde{M}.$$

Define $I(z) = \int_0^z (u^{1/q}/\phi(u))\,du$ so I is an increasing map from $[0, \infty)$ onto $[0, \infty)$ and therefore has in increasing inverse I^{-1}. Thus it follows that

$$y'(t) \leqslant I^{-1}(\tilde{M}) + 1/n \leqslant I^{-1}(\tilde{M}) + 1 \equiv M_1 \qquad \text{for} \quad t \in [0, 1].$$

We have thus shown that

$$1/n \leqslant y(t) \leqslant M_0, \qquad 1/n \leqslant y'(t) \leqslant M_1, \qquad t \in [0, 1]$$

for any solution y to $(5.4)_n$. Now assumption (5.5) implies that $-y''(t) \geqslant \psi(t)\,\eta(t)$ so

$$y'(t) \geqslant \int_t^1 \psi(s)\,\eta(s)\,ds \qquad \text{and} \qquad y(t) \geqslant t \int_t^1 \psi(s)\,\eta(s)\,ds + \int_0^t s\psi(s)\,\eta(s)\,ds.$$

Thus assumption (5.6) with the fact that g and ϕ are nonincreasing immediately yields the existence of a constant M_2 independent of n such that $\|y''\|_{L^r} \leqslant M_2$ for any solution y to $(5.4)_n$. ∎

THEOREM 5.2. *Suppose* (2.2), (2.3), (3.3), (5.5), *and* (5.6) *are satisfied. Then a* $C^1[0, 1] \cap C^2(0, 1)$ *solution of* (5.4) *exists.*

Proof. This follows from a slight modification of the proof of Theorem 5.1. ∎

Remark. Similarly we have corresponding results for the problems

$$\begin{gathered} y'' + \psi(t)\,f(t, y, y') = 0, \qquad 0 < t < 1 \\ y(1) = a > 0 \\ y'(0) = 0 \end{gathered}$$

and

$$\begin{gathered} y'' + \psi(t)\,f(t, y, y') = 0, \qquad 0 < t < 1 \\ y(1) = 0 \\ y'(0) = 0. \end{gathered}$$

It is of interest to conclude this paper by considering the case when $\psi \in C[0, 1]$ and f is singular *only* at $y' = 0$. In addition we extend the class of functions f that we examined previously so we obtain new and interesting existence theorems. In particular we look at

$$\begin{gathered} y'' + f(t, y, y') = 0, \qquad 0 < t < 1 \\ y(0) = a \geqslant 0 \\ y'(1) = b \geqslant 0. \end{gathered} \tag{5.7}$$

Assume to begin with that $b>0$ and in addition the following are satisfied:

f is continuous on $[0,1]\times(-\infty,\infty)\times(0,\infty)$ with $f>0$ on $[0,1]\times(0,\infty)\times(0,\infty)$ and $\lim_{p\to 0^+} f(t,y,p)=\infty$ for each $(t,y)\in[0,1]\times(-\infty,\infty)/\{0\}$. (5.8)

$0<f(t,y,p)\leqslant g(y)\,\phi(p)$ on $[0,1]\times(a,\infty)\times(0,\infty)$, where $\phi>0$ is continuous and nonincreasing on $(0,\infty)$ and g is continuous on $[a,\infty)$. (5.9)

There exist constants $A\geqslant 0$, $B\geqslant 0$, $0\leqslant q<1$, such that

$$\int_a^z g(u)\,du\leqslant\int_0^{Az^q+B}\frac{u}{\phi(u)}\,du \qquad \text{for all}\quad z\in[a,\infty). \tag{5.10}$$

THEOREM 5.3. *Let $b>0$ and suppose* (5.8), (5.9), *and* (5.10) *are satisfied. For $\lambda\in[0,1]$ consider the family of problems*

$$\begin{aligned} y''+\lambda f(t,y,y')&=0, \qquad 0<t<1\\ y(0)&=a\geqslant 0\\ y'(1)&=b>0. \end{aligned} \tag{5.7$_\lambda$}$$

Then there exist constants M_0, M_1, M_2 independent of λ such that for $t\in[0,1]$

$$a\leqslant y(t)\leqslant M_0, \qquad b\leqslant y'(t)\leqslant M_1, \qquad -M_2\leqslant y''(t)\leqslant 0$$

for each solution y to $(5.7)_\lambda$.

Proof. Let y be a solution to $(5.7)_\lambda$ and clearly $y(t)\geqslant a$, $y'(t)\geqslant b$, $y''(t)\leqslant 0$ for $t\in[0,1]$. We also have

$$-y''\leqslant g(y)\,\phi(y')\leqslant g(y)\,\phi(y'-b)$$

since ϕ is nonincreasing. Thus

$$\frac{-(y'-b)\,y''}{\phi(y'-b)}\leqslant g(y)(y'-b)\leqslant g(y)\,y'$$

and integration from t to 1 yields

$$\int_0^{y'(t)-b}\frac{u}{\phi(u)}\,du\leqslant\int_{y(t)}^{y(1)} g(u)\,du\leqslant\int_a^{y(1)} g(u)\,du.$$

Define $H(z)=\int_0^z (u/\phi(u))\,du$, so H is an increasing map from $[0, \infty)$ onto $[0, \infty)$ and therefore has an increasing inverse H^{-1}. So we have

$$y'(t) \leqslant H^{-1}\left(\int_a^{y(1)} g(u)\,du\right)+b. \tag{5.11}$$

Finally, integration from 0 to 1 yields

$$\begin{aligned} y(1) &\leqslant H^{-1}\left(\int_a^{y(1)} g(u)\,du\right)+b+a \\ &\leqslant A\{y(1)\}^q+B+b+a \end{aligned} \tag{5.12}$$

using assumption (5.10). Thus there exists a constant M_0 independent of λ such that $y(1) \leqslant M_0$. We have shown that $a \leqslant y(t) \leqslant M_0$ for any solution y to $(5.7)_\lambda$. In addition (5.11) implies

$$b \leqslant y'(t) \leqslant H^{-1}\left(\int_0^{M_0} g(u)\,du\right)+b \equiv M_1$$

for any solution y to $(5.7)_\lambda$. Now returning to the inequality $-y'' \leqslant g(y)\,\phi(y')$ we have for $t \in [0, 1]$

$$y''(t) \leqslant \phi(b) \sup_{[a, M_0]} g(y) \equiv M_2. \quad \blacksquare$$

We thus have immediately (via ideas in Theorem 2.2) the following theorem.

THEOREM 5.4. *Let* $b>0$ *and suppose* (5.8), (5.9), *and* (5.10) *are satisfied. Then a* $C^2[0, 1]$ *solution of* (5.7) *exists.*

To establish existence of a solution to (5.7) with $b=0$ we first consider for $n \in N^+$

$$\begin{gathered} y''+f(t, y, y')=0, \qquad 0<t<1 \\ y(0)=a \geqslant 0 \\ y'(1)=1/n. \end{gathered} \tag{5.13$_n$}$$

Suppose also (5.8), (5.9), (5.10), and (5.2) are satisfied. In addition assume there exists $r>1$ with

$$\int_0^1 \phi^r\left(\int_t^1 \eta(s)\,ds\right) dt<\infty. \tag{5.14}$$

Now let y be a solution to $(5.13)_n$. Then there exist constants M_0 and M_1 independent of n (see Eqs. (5.11) and (5.12)) such that

$$a \leqslant |y|_0 \leqslant M_0 \qquad \text{and} \qquad 1/n \leqslant |y'|_0 \leqslant M_1.$$

In addition

$$\|y''\|_{L^r} = \left\{\int_0^1 |y''(s)|^r\,ds\right\}^{1/r} \leqslant \left\{\int_0^1 g^r(y(t))\,\phi^r(y'(t))\,dt\right\}^{1/r}$$

$$\leqslant K\left\{\int_0^1 \phi^r\left(\int_t^1 \eta(s)\,ds\right)dt\right\}^{1/r} \equiv M_2$$

(independent of n) where $K = \sup_{[a, M_0]} g(y)$.

We thus have

THEOREM 5.5. *Let $b = 0$ and suppose* (5.2), (5.8), (5.9), (5.10), *and* (5.14) *are satisfied. Then a $C^1[0, 1] \cap C^2[0, 1)$ solution of* (5.7) *exists.*

Proof. This follows from a slight modification of the proof of Theorem 5.1. ∎

REFERENCES

1. L. E. BOBISUD AND D. O'REGAN, Existence of solutions to some singular initial value problems, *J. Math. Anal. Appl.* **133** (1988), 214–230.
2. L. E. BOBISUD, D. O'REGAN, AND W. D. ROYALTY, Singular boundary value problems, *Appl. Anal.* **23** (1986), 233–243.
3. L. E. BOBISUD, D. O'REGAN, AND W. D. ROYALTY, Solvability of some nonlinear boundary value problems, *Nonlinear Anal.* **12** (1988), 855–869.
4. A. CALLEGARI AND A. NACHMAN, Some singular nonlinear differential equations arising in boundary layer theory, *J. Math. Anal. Appl.* **64** (1978), 96–105.
5. A. CALLEGARI AND A. NACHMAN, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980), 275–281.
6. D. R. DUNNINGER AND J. C. KURTZ, Existence of solutions for some nonlinear singular boundary value problems, *J. Math. Anal. Appl.* **115** (1986), 396–405.
7. A. GRANAS, R. B. GUENTHER, AND J. W. LEE, A note on the Thomas Fermi equations, *Z. Angew. Math. Mech.* **60** (1980).
8. A. GRANAS, R. B. GUENTHER, AND J. W. LEE, Nonlinear boundary value problems for ordinary differential equations, *Dissertationes Math.* (1985).
9. C. D. LUNING AND W. L. PERRY, Positive solutions of negative exponent generalized Emden–Fowler boundary value problems, *SIAM J. Math. Anal.* **12** (1981), 874–879.
10. D. O'REGAN, Positive solutions to singular and nonsingular second order boundary value problems, *J. Math. Anal. Appl.* **142** (1989), 40–52.
11. S. D. TALIAFERRO, A nonlinear singular boundary value problem, *Nonlinear Anal.* **3** (1979), 897–904.
12. J. S. W. WANG, On the generalized Emden–Fowler equation, *SIAM Rev.* **17** (1975), 339–360.
13. G. IFFLAND, Positive solutions of a problem of Emden–Fowler type with a free boundary, *SIAM J. Math. Anal.* **18** (1987), 283–292.